

# Twist of Lie algebras by 6 dimensional subalgebra<sup>1</sup>

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## Abstract

A new non-standard deformation of all types of classical Lie algebras is constructed by means of Drinfel'd twist based on a six dimensional subalgebra. This is an extension of extended twists introduced by Kulish *et al.* For the algebra  $\mathcal{M}_3 \simeq so(3, 2)$ , a relation to a known non-standard deformation is discussed.

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# 1 Introduction

It is known that quantum algebras belong to one of two types of Hopf algebras: quasitriangular and triangular [1]. The  $q$ -deformed algebras by Drinfel'd and Jimbo belong to the first type. The algebras belong to the second type are called Jordanian quantum algebras or non-standard quantum algebras. The typical example of Jordanian quantum algebras is a deformation of  $sl(2)$  introduced by Ohn [2]. In general, triangular quantum algebras are obtained by applying Drinfel'd twist to Lie algebras [3]. The twisting that produces an isomorphic algebra to Ohn's one is found in [4, 5].

In this article, we discuss a new non-standard deformation of all types of classical Lie algebras, that can be regarded as an extension of non-standard quantum algebras introduced recently by Kulish *et al.* [6]. To make clear the novelty of our results and to fix notations, we briefly recall the definition of twisting and already known twisting for Lie algebras in the next section. The motivation of this study will be mentioned there. §3 is the main part of this article where the new non-standard deformation is introduced. As an physical example, we take an algebra of conformal transformations in (2+1) dimensional Minkowskian spacetime in §4. The equivalence of our result and another non-standard deformation by Herranz [7] is discussed. §5 is devoted to concluding remarks.

Drinfel'd developed the idea of twisting in his study of quasi-Hopf algebras, however, we restrict ourselves to ordinary Hopf algebras throughout this article.

## 2 Brief review of Drinfel'd twist

Let  $(\mathcal{H}_0, m_0, \Delta_0, \epsilon_0, S_0)$  be a Hopf algebra,  $m_0, \Delta_0, \epsilon_0$  and  $S_0$  denote product, coproduct, counit and antipode of  $\mathcal{H}_0$ , respectively. Suppose that an invertible element  $\mathcal{F} \in \mathcal{H}_0 \otimes \mathcal{H}_0$ , called twistor or twist element, satisfies

$$\mathcal{F}_{12}(\Delta_0 \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta_0)(\mathcal{F}), \quad (1)$$

$$(\epsilon_0 \otimes id)(\mathcal{F}) = (id \otimes \epsilon_0)(\mathcal{F}) = 1, \quad (2)$$

where  $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ ,  $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$  and  $id$  denotes the identity mapping as usual. Then  $(\mathcal{H}_0, m_0, \Delta, \epsilon_0, S)$  with

$$\Delta = \mathcal{F}\Delta_0\mathcal{F}^{-1}, \quad S = vS_0v^{-1}, \quad (3)$$

is also a Hopf algebra. We denote this new Hopf algebra by  $\mathcal{H}$ . The element  $v \in \mathcal{H}_0$  is obtained from the twistor. Writing  $\mathcal{F} = \sum_i f_i \otimes f^i$ , the element

$v$  is given by  $v = \sum_i f_i S_0(f^i)$ . Note that, in the new Hopf algebra  $\mathcal{H}$ , only coproduct and antipode are twisted, while product and coproduct are not changed. The universal  $R$ -matrix for  $\mathcal{H}$  is also obtained from the one for  $\mathcal{H}_0$ ,

$$\mathcal{R} = \mathcal{F}_{21} R_0 \mathcal{F}^{-1}, \quad (4)$$

where  $\mathcal{R}$  and  $\mathcal{R}_0$  are the universal  $R$ -matrices for  $\mathcal{H}$  and  $\mathcal{H}_0$ , respectively.

In the case of non-standard quantum algebras,  $\mathcal{H}_0$  is a Lie algebra (strictly speaking, universal enveloping algebra of a Lie algebra). Since the twisting does not change the product, non-standard quantum algebras have the same commutation relations as Lie algebras. However, they have deformed co-products and antipodes. The triangularity of non-standard algebras stems from the fact that  $\mathcal{R}_0$  for Lie algebras is given by  $1 \otimes 1$ . The advantages of non-standard quantum algebras are: (i) since they have undeformed commutation relations, we know their irreducible representations, (ii) explicit form of universal  $R$ -matrix is known. Combining (i) and (ii), we obtain matrix representations of universal  $R$ -matrix, this gives us further advantages: (iii) dual quantum groups are easily obtained by FRT-formalism [8], (iv) covariant differential calculus are easily obtained, and so on.

As is clear from the above discussion, once we obtain a explicit form of twistor, the rest steps of construction of non-standard quantum algebras and groups are automatically carried out. Note that the possibility of twisting for a Lie algebra is not unique, namely, a Lie algebra admits some twistors. Note also that non-standard algebras have some physical applications (see *e.g.* [9], [10] and [11]) and quantum groups are convenient tool to describe non-commutative spacetime. Therefore, we think that it is important to investigate possible twistors for Lie algebras. In the study of twisting, the following observation is useful: Let  $\mathcal{A}$  be a subalgebra of a Lie algebra  $\mathcal{H}_0$  and  $\mathcal{F}$  is a twistor for the subalgebra  $\mathcal{A}$ . Then, the twistor  $\mathcal{F}$  can be regarded as a twistor for  $\mathcal{H}_0$  and we can twist  $\mathcal{H}_0$  with this twistor. In the following, we give five examples of twistors classified according to the subalgebra, then we discuss an extension of one of them.

Probably, the most wellknown twistors are the so-called Reshetikhin twist [12] and Jordanian twist [4, 5]. In the Reshetikhin twist, the Cartan subalgebra is used as the subalgebra. Thus any Lie algebras of rank  $\geq 2$  admit Reshetikhin twist. On the other hand, the subalgebra for Jordanian twist is the Borel subalgebra:  $\{H, E \mid [H, E] = 2E\}$ . The explicit form of twistor is given by

$$\mathcal{F}_J = \exp\left(-\frac{1}{2}H \otimes \sigma\right), \quad \sigma = -\ln(1 - zE), \quad (5)$$

where  $z$  is a deformation parameter. We denote a deformation parameter by  $z$  throughout this article and  $z = 0$  corresponds to undeformed limit.

Three nontrivial extensions of Jordanian twist were considered very recently [6, 13, 14]. In [6], the Borel subalgebra is extended to four dimensional one that is a semidirect sum of Borel subalgebra and two additional elements  $A, B$ . Let  $\mathcal{A}_E = \{H, E, A, B\}$  be the subalgebra subject to the relations

$$\begin{aligned} [H, E] &= \delta E, & [H, A] &= \alpha A, & [H, B] &= \beta B, \\ [A, B] &= \gamma E, & [E, A] &= [E, B] = 0, & \alpha + \beta &= \delta. \end{aligned} \quad (6)$$

Then the following is a twistor

$$\mathcal{F}_E = \exp(A \otimes B e^{-\beta\sigma/\delta}) \mathcal{F}_J. \quad (7)$$

This twisting is called extended (Jordanian) twist. It is verified that all types of classical Lie algebra have the subalgebra  $\mathcal{A}_E$ . We use different convention from [6] for the definition of  $\sigma$ , but this may not cause any confusion. The extension considered in [13] is certain limits of extended twists and called peripheric extended twists. The extended twists have nontrivial limits for  $\alpha \rightarrow 0$  or  $\beta \rightarrow 0$ . One can verify that the peripheric extended twists are applicable to inhomogeneous Lie algebras  $isu(n)$  and  $iso(n)$ . In [14], regular injections  $\mathcal{A}_p \subset \mathcal{A}_{p-1} \subset \dots \subset \mathcal{A}_1 \subset \mathcal{A}_0$  of Lie algebras and twistors of extended twists are considered. It is shown that a product of twistors of extended twists corresponding to each subsets  $\mathcal{A}_k (k = 0, 1, \dots, p)$  can produce a new twistor for the following sequences:

$$\begin{aligned} sl(n) &\supset sl(n-2) \supset \dots \supset sl(n-2k) \supset \dots, \\ so(2n) &\supset so(2n-4) \supset \dots \supset so(2n-4k) \supset \dots, \\ so(2n+1) &\supset \dots \supset so(2n-4k+1) \supset \dots \end{aligned}$$

The case of  $sp(n)$  is also considered in [14], though the situation is different from other classical Lie algebras.

All twistors mentioned above except Rshetikhin twist have common properties: (i) twisted coproduct for  $\sigma$  is primitive, that is,  $\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$ , (ii) the twistors are factorizable, that is, they satisfy the relations

$$(\Delta_0 \otimes id)(\mathcal{F}) = \mathcal{F}_{13} \mathcal{F}_{23}, \quad (id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12} \mathcal{F}_{13}. \quad (8)$$

These relations guarantee that the  $\mathcal{F}$  satisfies the condition (1).

In the next section, we consider a further extension of extended twists in such a way that the four dimensional subalgebra  $\mathcal{A}_E$  is replaced with six dimensional one.

### 3 New twisting for classical Lie algebras

Let us consider an algebra  $\mathcal{A}$  of six elements  $H_i, E_i, A, B$  ( $i = 1, 2$ ) satisfying

$$\begin{aligned} [H_i, E_i] &= 2E_i, \\ [H_1, H_2] &= [E_1, E_2] = [H_1, E_2] = [H_2, E_1] = 0, \\ [H_1, A] &= -A, \quad [H_1, B] = B, \quad [H_2, A] = A, [H_2, B] = B, \\ [A, E_1] &= 2B, \quad [A, E_2] = 0, \quad [E_i, B] = 0, \quad [A, B] = E_2. \end{aligned} \quad (9)$$

The four elements  $\{H_2, E_2, A, B\}$  form the subalgebra  $\mathcal{A}_E$  of extended twists ( $\alpha = \beta = \gamma = 1$ ). The additional elements  $H_1, E_1$  form a Borel subalgebra, thus the algebra  $\mathcal{A}$  is a semidirect sum of  $\mathcal{A}_E$  and an extra Borel subalgebra. We can also regard the algebra  $\mathcal{A}$  as a semidirect sum of  $\{A, B\}$  and a direct sum of two Borel subalgebras

$$\mathcal{A} = (\{H_1, E_1\} \oplus \{H_2, E_2\}) \ltimes \{A, B\}.$$

The following invertible element  $\mathcal{F}$  satisfies the definition of twistor.

$$\mathcal{F} = \exp\left(-\frac{1}{2}H_1 \otimes \sigma_1\right) \exp(-zA \otimes B e^{\frac{1}{2}\sigma_2}) \exp\left(-\frac{1}{2}H_2 \otimes \sigma_2\right), \quad (10)$$

where

$$\sigma_1 = -\ln(1 - z(E_1 + zB^2 e^{\sigma_2})), \quad \sigma_2 = -\ln(1 - zE_2). \quad (11)$$

The two factors from the right are a extended twist and left most factor does not commute with the rest part of  $\mathcal{F}$ . Therefore this is a nontrivial extension of extended twists. The twistor (10) has similar properties as extended twists. Namely, the twisted coproducts for  $\sigma_i$  are primitive

$$\Delta(\sigma_i) = \sigma_i \otimes 1 + 1 \otimes \sigma_i, \quad i = 1, 2, \quad (12)$$

and the  $\mathcal{F}$  is factorizable (see eq.(8)).

To prove the above statements, we first calculate the twisted coproducts of the elements of  $\mathcal{A}$  by the twistor (10). They are given by

$$\begin{aligned} \Delta(H_1) &= H_1 \otimes e^{\sigma_1} + 1 \otimes H_1, \\ \Delta(E_1) &= E_1 \otimes e^{-\sigma_1} + 1 \otimes E_1 - 2zB \otimes \rho e^{-(\sigma_1+\sigma_2)/2} + z^2 E_2 \otimes \rho^2 e^{-\sigma_2}, \\ \Delta(H_2) &= H_2 \otimes e^{\sigma_2} + 1 \otimes H_2 + 2zA \otimes \rho e^{(\sigma_1+\sigma_2)/2} + z^2 H_1 \otimes \rho^2 e^{\sigma_1}, \\ \Delta(E_2) &= E_2 \otimes e^{-\sigma_2} + 1 \otimes E_2, \\ \Delta(A) &= A \otimes e^{(\sigma_1+\sigma_2)/2} + 1 \otimes A + zH_1 \otimes (B + zE_2 \rho) e^{\sigma_1}, \\ \Delta(B) &= B \otimes e^{-(\sigma_1+\sigma_2)/2} + e^{-\sigma_2} \otimes B, \end{aligned} \quad (13)$$

where  $\rho = Be^{\sigma_2}$ . With these coproducts, we can verify that the  $\sigma_i$ 's are primitive. It follows that the twistor (10) satisfies the factorizable relations (8). Thus we have proved that the  $\mathcal{F}$  satisfies the condition (1). The condition (2) is easily verified by noticing that  $\epsilon_0(X) = 0$  for all elements of any Lie algebras.

The universal  $R$ -matrix of the twisted Lie algebra is given by

$$\begin{aligned} R = \mathcal{F}_{21}\mathcal{F}^{-1} &= \exp(-\frac{1}{2}\sigma_1 \otimes H_1) \exp(-z\rho \otimes A) \exp(-\frac{1}{2}\sigma_2 \otimes H_2) \\ &\times \exp(\frac{1}{2}H_2 \otimes \sigma_2) \exp(zA \otimes \rho) \exp(\frac{1}{2}H_1 \otimes \sigma_1). \end{aligned} \quad (14)$$

One can easily write down the corresponding classical  $r$ -matrix that solves the classical Yang-Baxter equation by keeping up to the first order of the deformation parameter

$$r = \frac{1}{2}H_1 \wedge E_1 + \frac{1}{2}H_2 \wedge E_2 + A \wedge B. \quad (15)$$

We next show that all types of classical Lie algebras have the six dimensional subalgebra  $\mathcal{A}$ , namely, we obtain new non-standard quantum algebras. For  $sl(n)$ , it is convenient to work with canonical basis

$$[E_{ab}, E_{cd}] = E_{ad}\delta_{bc} - E_{cb}\delta_{ad}, \quad a, b, c, d = 1, \dots, n. \quad (16)$$

In this case, the six dimensional subalgebra  $\mathcal{A}$  is found for  $n \geq 4$

$$\begin{aligned} H_1 &= \sum_{n/2 \geq k \geq 2} (E_{kk} - E_{n-k+1, n-k+1}), \quad E_1 = \sum_{n/2 \geq k \geq 2} E_{k, n-k+1}, \\ H_2 &= E_{11} - E_{nn}, \quad E_2 = E_{1n}, \\ A &= 2 \sum_{n/2 \geq k \geq 2} b^{1, n-k+1} E_{1k} - 2 \sum_{n-1 \geq \lambda > n/2} b^{n-\lambda+1, n} E_{\lambda n}, \\ B &= \sum_{n/2 \geq k \geq 2} b^{k, n} E_{kn} + \sum_{n-1 \geq \lambda > n/2} b^{1, \lambda} E_{1\lambda}, \end{aligned} \quad (17)$$

where the complex coefficients  $b^{a,b}$ 's have to satisfy

$$4 \sum_{n/2 \geq k \geq 2} b^{1, n-k+1} b^{k, n} = 1. \quad (18)$$

This condition on the coefficients stems from the commutation relation  $[A, B] = E_2$ . Other commutation relations are hold for any values of  $b^{a,b}$ .

We also use the canonical basis for  $so(n)$ .

$$\begin{aligned} [Y_{ab}, Y_{cd}] &= i(Y_{ad}\delta_{bc} + Y_{bc}\delta_{ad} - Y_{ac}\delta_{bd} - Y_{bd}\delta_{ac}), \\ Y_{ab} &= -Y_{ba}, \quad a, b, c, d = 1, \dots, n. \end{aligned} \quad (19)$$

In this case, the  $\mathcal{A}$  is found for  $n \geq 5$ .

$$\begin{aligned} H_1 &= Y_{1n} + Y_{1\ n-1}, & E_1 &= Y_{1\ n-1} - Y_{2n} + iY_{12} + iY_{n-1\ n}, \\ H_2 &= Y_{1n} - Y_{1\ n-1}, & E_2 &= \frac{1}{2}(Y_{1\ n-1} + Y_{2n} - iY_{12} + iY_{n-1\ n}), \\ A &= \sum_{k=3}^{n-2} a^k (Y_{n-1\ k} - iY_{2k}), & B &= \sum_{k=3}^{n-2} a^k (Y_{kn} - iY_{1k}). \end{aligned} \quad (20)$$

The commutation relation  $[A, B] = E_2$  imposes a condition on the coefficients  $a^k$

$$2 \sum_{k=3}^{n-2} (a^k)^2 = 1, \quad a^k \in \mathbf{C}. \quad (21)$$

For  $sp(2n)$ , the six dimensional subalgebra  $\mathcal{A}$  is found for  $n \geq 2$ . In terms of the canonical basis

$$\begin{aligned} [Z_{ab}, Z_{cd}] &= \text{sign}(bc)(Z_{ad}\delta_{bc} + Z_{-b\ -c}\delta_{ad} + Z_{a\ -c}\delta_{-b\ d} + Z_{-b\ d}\delta_{c\ -a}), \\ Z_{ab} &= -\text{sign}(ab)Z_{-b\ -a}, \quad a, b = \pm 1, \dots, \pm n, \end{aligned} \quad (22)$$

the elements of  $\mathcal{A}$  are given by

$$\begin{aligned} H_1 &= \sum_{k=2}^n Z_{kk}, & E_1 &= \sum_{k=2}^n Z_{k\ -k}, & H_2 &= Z_{11}, & E_2 &= Z_{1\ -1}, \\ A &= \sum_{k=2}^n a^k Z_{1k}, & B &= \sum_{k=2}^n a^k Z_{k\ -1}. \end{aligned} \quad (23)$$

A condition on the coefficients  $a^k$ 's are obtained same way as  $sl(n)$  and  $so(n)$

$$\sum_{k=2}^n (a^k)^2 = 1, \quad a^k \in \mathbf{C}. \quad (24)$$

We have seen that all types of classical Lie algebras can be twisted by the six dimensional subalgebra  $\mathcal{A}$ . Other combinations of elements of Lie algebras could realize the subalgebra  $\mathcal{A}$ . An appropriate choice could be found when physical applications of twisted algebras are considered.

## 4 Conformal algebra of (2+1)-dimensional space-time

In this section, we apply the twisting by  $\mathcal{A}$  to the algebra  $\mathcal{M}_3 \simeq so(3, 2)$ : the algebra of conformal transformations in the (2+1)-dimensional Minkowskian

spacetime. We consider the following basis of  $\mathcal{M}_3 = \{J, P_\mu, K_i, C_\mu, D\}$  where  $\mu = 0, 1, 2$ ,  $i = 1, 2$  and  $J$  is a generator of rotations,  $P_0$  time translations,  $P_i$  space translations,  $K_i$  boosts,  $C_\mu$  special conformal transformations, and  $D$  dilatations. The commutation relations of the algebra are given in eq.(1.1) of [7]. The subalgebra  $\mathcal{A}$  is given by

$$\begin{aligned} H_1 &= D + K_1, & E_1 &= P_0 + P_1, & H_2 &= D - K_1, \\ E_2 &= P_0 - P_1, & A &= K_2 + J, & B &= P_2. \end{aligned} \quad (25)$$

Therefore, we have obtained a twisted  $\mathcal{M}_3$  that has undeformed commutation relations but deformed coproducts. It may be remarkable that no generators of conformal transformations appear in the algebra  $\mathcal{A}$ .

On the other hand, another non-standard deformation of  $\mathcal{M}_3$  is considered in [7]. We denote it by  $U_z(\mathcal{M}_3)$  and its generators by  $\tilde{J}, \tilde{P}_\mu$  etc. The algebra  $U_z(\mathcal{M}_3)$  is defined by deformed commutation relations and deformed coproducts (eqs.(3.3)-(3.6) of [7]). The universal  $R$ -matrix for  $U_z(\mathcal{M}_3)$  is not known. We here give two observations that imply an equivalence between our twisted  $\mathcal{M}_3$  and  $U_z(\mathcal{M}_3)$ . First, there exist an invertible mapping between the Weyl subalgebras of twisted  $\mathcal{M}_3$  and  $U_z(\mathcal{M}_3)$

$$\begin{aligned} \tilde{D} &= D, & \tilde{J} &= J, & \tilde{K}_i &= K_i, \\ \tilde{P}_0 &= \frac{1}{2z}(\sigma_1 + \sigma_2), & \tilde{P}_1 &= \frac{1}{2z}(\sigma_1 - \sigma_2), & \tilde{P}_2 &= P_2 e^{\sigma_2}. \end{aligned} \quad (26)$$

By this mapping, both undeformed commutation relations and deformed coproducts of twisted  $\mathcal{M}_3$  are transformed into the corresponding ones of  $U_z(\mathcal{M}_3)$ . The mapping for  $C_\mu$  is not known. Probably, it has quite complicated form. Second, twisted  $\mathcal{M}_3$  and  $U_z(\mathcal{M}_3)$  have the same classical  $r$ -matrix

$$r = z(D \wedge P_0 + K_1 \wedge P_1 + (K_2 + J) \wedge P_2). \quad (27)$$

This classical  $r$ -matrix (and more general form with two deformation parameters) was firstly discussed in [15]. If the equivalence is true, by using (26) we can easily write down the universal  $R$ -matrix for  $U_z(\mathcal{M}_3)$  in terms of its generators.

Two contractions of  $\mathcal{M}_3$ , that produce conformal algebras of the (2+1) Galilean and the Carroll spacetime, are also considered in [7] and corresponding contractions for  $U_z(\mathcal{M}_3)$  are analyzed. It is impossible to apply the contractions to our twistor, since the contracted algebras do not have the six dimensional subalgebra  $\mathcal{A}$ . However, we can find twistors for each contracted algebras and these twistors give the same classical  $r$ -matrices as [7]. For the conformal algebra of the Galilean spacetime,

$$\mathcal{F} = \exp(-zK_1 \otimes P_1) \exp(-zK_2 \otimes P_2). \quad (28)$$



The commutation relations of this algebra are given in eq.(1.4) of [7]. Two factors of (28) commute each other. For the conformal algebra of the Carroll spacetime

$$\mathcal{F} = \exp(-D \otimes \sigma) \exp(-zK_1 \otimes P_1 e^\sigma) \exp(-zK_2 \otimes P_2 e^\sigma), \quad (29)$$

where  $\sigma = -\ln(1 - zP_0)$ . The commutation relations of this algebra are eq.(1.5) of [7]. Three factors of (29) commute one another.

## 5 Concluding remarks

In this article, we have shown a new twistor that is an extension of the extended twists is applicable to all types of classical Lie algebras. Consequently new non-standard deformations of  $sl(n)$ ,  $so(n)$  and  $sp(2n)$  was obtained. When the twistor is applied to the algebra  $\mathcal{M}_3$ , the obtained algebra seems to be equivalent to a non-standard deformation of  $\mathcal{M}_3$  by Herranz.

It is natural to ask whether the peripheric extended twists have a similar extension. Since the twists discussed in §3 do not contain free parameters ( $\alpha, \beta$  and  $\gamma$  of the extended twists), we can not repeat the same discussion as [13]. However, it turns out that the peripheric extended twists have an extension [16] where a five dimensional subalgebra is used instead of the six dimensional one. This extension of peripheric extended twists is appropriate to deform inhomogeneous Lie algebras.

Finally, we mention a physical application of the twisted algebras. It is known that symmetries of free massless Klein-Gordon equation are given by conformal algebras [17]. We mean, by symmetry, transformations of solutions into other solutions. If a difference analogue of Klein-Gordon equation on the uniform lattice is considered in (2+1)-dimensional spacetime, the twisted  $\mathcal{M}_3$  can be regarded as its symmetry algebra. This will be discussed elsewhere. The similar situation was considered for (1+1)-dimensional free Schrödinger equation [10, 11].

After finishing this article, a paper [18] that obtains the same result for  $so(5)$  was submitted to the preprint archive.

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